

ON (a, b) PAIRS IN RANDOM FIBONACCI SEQUENCES

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ABSTRACT. We study the random Fibonacci tree, which is an infinite binary tree with non-negative numbers at each node defined as follows. The root consists of the number 1 with a single child also the number 1. Then we define the tree recursively in the following way: if x is the parent of y , then y has two children, namely $|x - y|$ and $x + y$. This tree was studied by Benoit Rittaud [6] who proved that any pair of integers a, b that are coprime occur as a parent-child pair infinitely often. We extend his results by determining the probability that a random infinite path in this tree contains exactly one pair $(1, 1)$, that being at the root of the tree. Also, we give tight upper and lower bounds on the number of occurrences of any specific coprime pair (a, b) at any specific level down the tree.

1. INTRODUCTION

The Fibonacci sequence recursively defined by $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$ has been generalised in several ways. In 2000, Divakur Viswanath studied random Fibonacci sequences given by $t_1 = t_2 = 1$ and $t_n = \pm t_{n-1} \pm t_{n-2}$ for all $n \geq 3$ where the sign choosing of \pm is independent for each one and each \pm is replaced with $+$ or $-$ with probability $1/2$ for each. Viswanath proved that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|t_n|} = 1.13198824\dots$$

with probability 1 [8]. An exact value is still unknown.

In 2006, Jeffrey McGowan and Eran Makover used the formalism of trees to give a simpler proof of Viswanath's result to evaluate the growth of the average value of the n th term [5]. More precisely, they proved the following theorem:

$$1.12095 \leq \sqrt[n]{E(|t_n|)} \leq 1.23375$$

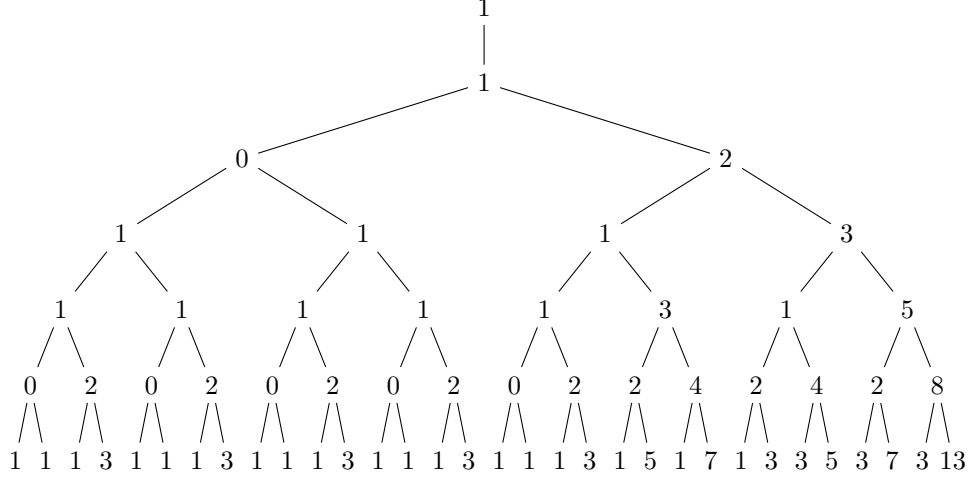
where $E(|t_n|)$ is the expected value of the n th term of the sequence.

In 2007, Benoit Rittaud used McGowan and Makover's idea of trees to construct full binary Fibonacci trees in the following way. The root, which is at the top, consists of a number g_0 with a single child g_1 , with at least one of these two values not being 0. Then he defined the tree recursively in the following way: if x is the parent of y , then y has two children, namely $|x - y|$ on the left branch and $x + y$ on the right branch. Rittaud denoted this tree as $\mathbf{T}_{(g_0, g_1)}$. Letting m_n denote the mean value of the 2^{n-1} values n branches down the tree, Rittaud proved that, independent of the choices for g_0 and g_1 ,

$$\lim_{n \rightarrow \infty} \frac{m_{n+1}}{m_n} = \alpha - 1 \approx 1.20556943$$

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FIGURE 1. Top of Tree $T_{(1,1)}$

where α is the only real number, satisfying $\alpha^3 = 2\alpha^2 + 1$ [6]. Rittaud also showed that when $g_0 = g_1 = 1$, the resulting tree will have the property that an ordered pair of natural numbers (a, b) will occur on a single branch of this tree with a being the parent of b if and only if $\gcd(a, b) = 1$ and that such any such pair occurs infinitely many times. In particular, this is true for the pair $(1, 1)$.

Here we also consider $\mathbf{T}_{(1,1)}$. The top seven levels are given in Figure 1. We extend Rittaud's result on (a, b) pairs occurring infinitely often by giving tight bounds on the number of such pairs at any level down the tree. For all $n \in \mathbb{N} \cup \{0\}$, we observe the following. If you go $3n$ branches down the tree, you arrive at coprime pairs (a, b) where both a and b are odd. If you go $3n + 1$ branches down the tree, you arrive at coprime pairs where a is odd and b is even. Finally, if you go $3n + 2$ branches down the tree, you arrive at coprime pairs where a is even and b is odd. This leads to the following definition.

Definition 1.1. We denote by $A_{(a,b)}(n)$ the number of (a, b) pairs that are found by going $3n + m$ branches down the Fibonacci tree. Here we have $m = 0$ if a and b are odd, $m = 1$ if a is odd and b is even, and $m = 2$ if a is even and b is odd. For any specific pair (a, b) when we speak of the pairs (a, b) at level n in the Fibonacci tree we mean the pairs that occur $3n + m$ branches down the tree.

Note 1.1. Since Sections 2, 3, 4, and 6 deal exclusively with $(1, 1)$ pairs, we will use $A(n)$ in place of $A_{(1,1)}(n)$ for simplicity of notation in those sections.

In [6] Rittaud introduced the concept of what he refers to as a single 0-walk, which is a walk that goes $3n + 2$ branches down the tree and traverses a node with number 0 at the end and does not traverse 0 before this last node. He showed that the number of these walks $3n + 2$ branches down the tree is

$$\frac{1}{2n+1} \binom{3n}{n}.$$

In each such path, however, since the last node is 0, it can easily be seen that the third last and second last nodes form a $(1, 1)$ pair at level n . This leads to the following definition.

Definition 1.2. We denote by $B(n)$ the number of $(1, 1)$ pairs that are found by going $3n$ branches down the Fibonacci tree such that 0 is never traverse in any of the paths to these $(1, 1)$ pairs. Also, we denote by $S(n)$ the number of $(1, 1)$ pairs that are found by going $3n$ branches down the Fibonacci tree such that another $(1, 1)$ is never traverse in any of the paths to these $(1, 1)$ pairs, except for at the starting root. We call these $S(n)$ pairs primitive.

We have

$$(1.1) \quad S(n) \leq B(n) = \frac{1}{2n+1} \binom{3n}{n} \leq A(n).$$

Since the path to any $(1, 1)$ that isn't primitive must go through an intermediate primitive pair $(1, 1)$, we have the formula

$$(1.2) \quad A(n) = \sum_{i=0}^{n-1} A(i)S(n-i).$$

It is also worth noting that the $(1, 1)$ pairs that the function $B(n)$ counts is exactly the $(1, 1)$ pairs whose paths begin with a right branch and have the property that the path takes a right branch immediately after every intermediate $(1, 1)$ pair traverse. Any path that doesn't have this property would have to traverse an intermediate 0, contradicting our definition of $B(n)$. Conversely, every path that has this property cannot traverse any node that has a 0 for the only way to traverse a 0 is to take an immediate left branch after a pair $(1, 1)$.

Using this fact of the pairs that $B(n)$ counts, we also have the formula

$$(1.3) \quad B(n+1) = B(n) + \sum_{i=0}^{n-2} B(i)S(n-i)$$

We'll be using Equations (1.2) and (1.3) in the rest of the paper.

We first consider how often can we avoid the pair $(1, 1)$. In Section 2 we prove:

Theorem 1.1. Choose a path down the tree randomly, starting with the root $(1, 1)$, with the probability of choosing a right branch be p and left branch $1 - p$. Then the probability the path does not contain any $(1, 1)$ pair except at the root is 0 if $p \leq 1/3$ and is

$$\frac{3p - 2 + \sqrt{4p - 3p^2}}{2}$$

if $p > 1/3$.

In the other direction, precise asymptotics for $A(n)$ are developed in Section 6. Namely, we prove that

Theorem 1.2.

$$A(n) = \frac{243 \cdot 6.75^n}{4\sqrt{3\pi}n^{3/2}} - \frac{337041 \cdot 6.75^n}{288\sqrt{3\pi}n^{5/2}} + O\left(\frac{6.75^n}{n^{7/2}}\right)$$

with the implicit constant of the error term always lying between 0 and $\frac{1348164}{36\sqrt{3}\pi}$ for all $n \in \mathbb{N}$. That is, for sufficiently large n we have

$$\left(\frac{243 \cdot 6.75^n}{4\sqrt{3}\pi n^{3/2}}\right) \left(1 - \frac{1387}{72n}\right) < A(n) < \left(\frac{243 \cdot 6.75^n}{4\sqrt{3}\pi n^{3/2}}\right) \left(1 - \frac{1387}{72n} + \frac{5548}{9n^2}\right).$$

Also, in Section 7, we develop precise asymptotics for $A_{(a,b)}$ for all coprime pairs (a, b) . Namely, we prove that

Theorem 1.3. *For all coprime pairs (a, b) , there exists an explicitly computable positive constant $C_{(a,b)}$ and an asymptotically estimatable rational constant $D_{(a,b)}$ such that*

$$A_{(a,b)}(n) = \frac{C_{(a,b)} \cdot 6.75^n}{n^{3/2}} + \frac{C_{(a,b)} D_{(a,b)} 6.75^n}{n^{5/2}} + O\left(\frac{6.75^n}{n^{7/2}}\right)$$

where the implied constant in the error term depends upon the number of branches in the shortest path from the root $(1, 1)$ to the pair (a, b) .

The paper is divided up as follows. Theorem 1.1 is given in Section 2.

In Sections 3 and 4 we develop asymptotic formulas for $S(n)$ and $B(n)$, and a weak upper bound for $A(n)$, which will be used in the proof of Theorem 1.2. In Section 5 we develop some preliminary results for other coprime pairs, which are used in both the proofs of Theorems 1.2 and 1.3. Sections 6 and 7 provide the proofs of Theorems 1.2 and 1.3 respectively.

The last section, Section 8, discusses some open questions related to this research.

Note 1.2. *Whenever we write 6.75 in this paper, we mean this exact value of $\frac{27}{4}$.*

Notation 1.1. *Suppose we have two functions $f, g : \mathbb{N} \rightarrow \mathbb{R}$. In the rest of the paper, we use the notation*

$$f(n) \sim g(n)$$

to mean that

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1.$$

2. RANDOM PATHS IN THE TREE

In this section we will consider the problem of how often we expect to find a random infinite path that never traverses the pair $(1, 1)$. Here we prove Theorem 1.1. We first need some preliminary lemmas.

Lemma 2.1. *Take a path (a, b) to another pair (c, d) and suppose the numbers on the nodes encountered in order are $a, b, a_1, a_2, \dots, a_n, c, d$. Then reversing the order of this list of numbers to get $d, c, a_n, \dots, a_2, a_1, b, a$ gives a path from (d, c) to (b, a) .*

Proof. We only have to show that if a_1, a_2, a_3 occur in the given path, then a_3, a_2, a_1 can consecutively occur in a path in that order. We have either $a_3 = a_1 + a_2$ or $a_3 = |a_2 - a_1|$. In the first case, we have $a_1 = |a_2 - a_3|$ giving us our result. In the second case, we either have $a_3 = a_2 - a_1$, giving us $a_1 = a_2 - a_3$, or $a_3 = a_1 - a_2$, giving us $a_1 = a_2 + a_3$. \square

Lemma 2.2. *The shortest path from a non- $(1, 1)$ pair to a $(1, 1)$ pair is characterised as a series of left branches with no right branches.*

Proof. By Lemma 2.1, the shortest path is obtained by going backwards on the shortest path from $(1, 1)$ to given pair non- $(1, 1)$ pair (a, b) . In [6, Corollary 5.1] states this shortest path must have the property that for any pair (c, d) occurring in the walk, the parent of c is $|c - d|$. Thus the shortest path from (a, b) to $(1, 1)$ must have the property that for any pair (c, d) occurring in the walk, the child of d is $|c - d|$, a choice of a left branch. Thus the shortest path must contain no right branches. \square

Lemma 2.3. *Take a non- $(1, 1)$ pair (a, b) appearing in the tree and suppose it takes n left branches to reach the nearest $(1, 1)$. Then it will take $n - 1$ left branches for $(b, |b - a|)$ to reach $(1, 1)$. The shortest path for $(b, a + b)$ to reach $(1, 1)$, however, will have length $n + 2$.*

Proof. The shortest path from $(b, a + b)$ to $(1, 1)$ will consist only of left branches by Lemma 2.2. Thus the path will start in the following way:

$$b, a + b, a, b$$

Thus after two left branches we're back at (a, b) from which we will take n left branches. Thus the shortest path from $(b, a + b)$ to $(1, 1)$ is of length $n + 2$. \square

Definition 2.1. *We will call a pair (a, b) occurring in a specific place in the tree primitive if and only if the path to that specific place contains no (a, b) pairs before the final pair that is (a, b) .*

Proposition 2.1. *The paths to all the primitive $(1, 1)$ s in the tree at a level greater than 1 can be characterised as starting with a right branch, having twice as many left branches as right branches, and the path to any intermediate pair must have the number of right branches be strictly more than twice the number of left branches.*

Proof. The fact that the first branch has to be a right branch follows from the observation that a left branch will just lead to all $(1, 1)$ pairs at level 1. The first right branch consists of the pair $(1, 2)$. From here it takes two left branches to reach the nearest $(1, 1)$ pair. Suppose we have path from this $(1, 2)$ to a primitive $(1, 1)$. In going through such a path, we consider each intermediate pair and consider how long the shortest path is from an intermediate pair to $(1, 1)$ (which may or may not deviate from our original path).

Suppose we were to take all these pairs in order and replace each pair with the shortest path from the given pair to $(1, 1)$. Then we would get a sequence of integers starting with 2 (since this is the shortest path length from $(1, 2)$ to $(1, 1)$), each successive element is obtained by adding 2 to the previous element (representing going down a right branch) or subtracting 1 from the previous element (representing going down a left branch) by Lemma 2.3. Finally, all integers in the sequence will be positive, with the exception of the last being 0 since we're dealing with a path to $(1, 1)$ and the shortest path from $(1, 1)$ to $(1, 1)$ is of length 0.

One property of such a sequence is that if r is the number of times you add 2, then $2r + 2$ must be the number of times you subtract 1. Moreover anywhere in the sequence except at the last element if s is the number of times you added 2 up to that point, then you cannot have subtracted 1 more than $2s + 1$ times. Moreover, it is easily seen that if we have a finite integer sequence starting with 2 with the above rules in play, then all the elements in the sequence will be positive except for the last one, which will be a 0.

Thus the paths to all the primitive $(1, 1)$ s in the tree at a level greater than 1 can be characterised as starting with a right branch, having twice as many left branches as right branches, and the path to any intermediate pair must have the number of right branches be strictly more than twice the number of left branches. \square

Proof of Theorem 1.1. Each path not containing a pair $(1, 1)$ except at the root must begin with a right branch. From there each desirable path can correspond to an infinite positive integer sequence, each number denoting the shortest path to a $(1, 1)$ pair from a specific pair in the given path exactly like in the proof of Proposition 2.1. Thus we can consider the problem of having random integer sequences beginning with 2 and either adding 2 or subtracting 1 to get the next number. We want to know the probability of such a sequence having all of its terms be positive.

For each $n \in \mathbb{N} \cup \{0\}$ let us denote the probability of starting a sequence with n and applying the above rules and eventually traversing 0 with $P(n)$. Thus we have the recurrence

$$(2.1) \quad P(n) = (1 - p)P(n - 1) + pP(n + 2), n \neq 0$$

with $P(0) = 1$ since the successor n in the sequence will either be $n - 1$ with probability $1 - p$ or $n + 2$ with probability p .

We can prove that there exists constants A, B , and C such that for all $n \in \mathbb{N} \cup \{0\}$, we have

$$(2.2) \quad P(n) = A + Br_1^n + Cr_2^n$$

where

$$r_1 = \frac{-1 + \sqrt{4/p - 3}}{2} \text{ and } r_2 = \frac{-1 - \sqrt{4/p - 3}}{2}.$$

From (2.1), we obtain for $n \geq 3$ that

$$P(n) = \frac{P(n - 2)}{p} - \frac{P(n - 3)(1 - p)}{p}.$$

By our choice of A, B, C we have the above holds for $n = 0, 1, 2$ automatically. We can then show by induction that it holds for all $n \in \mathbb{N} \cup \{0\}$ by a standard linear recurrence argument using the facts that

$$r_1^n = \frac{r_1^{n-2}}{p} - \frac{r_1^{n-3}(1 - p)}{p} \text{ and } r_2^n = \frac{r_2^{n-2}}{p} - \frac{r_2^{n-3}(1 - p)}{p}$$

for all $n \in \mathbb{N} \cup \{0\}$.

We now split into cases.

Case 1. $p < 1/3$

We can work out that $r_1 > 1$ and $r_2 < -2$. Therefore, if $B \neq 0$ or $C \neq 0$, then by (2.2) we have

$$\limsup_{n \rightarrow \infty} P(n) = \infty,$$

a contradiction since we want $0 \leq P(n) \leq 1$ for all $n \in \mathbb{N}$. Therefore $B = C = 0$ and since $P(0) = 1$, we have

$$P(n) = 1$$

for all $n \in \mathbb{N} \cup \{0\}$. Therefore have that the probability of a random path not traversing a pair $(1, 1)$ except at the root is

$$p(1 - P(2)) = p(1 - 1) = 0.$$

Case 2. $p = 1/3$

We can work out that $r_1 = 1$ and $r_2 = -2$ so that from (2.1) we get

$$P(n) = A + B + C(-2)^n.$$

If $C \neq 0$, then we have

$$\limsup_{n \rightarrow \infty} P(n) = \infty,$$

a contradiction since we want $0 \leq P(n) \leq 1$. Therefore $C = 0$ and since $P(0) = 1$, we have

$$P(n) = 1$$

for all $n \in \mathbb{N} \cup \{0\}$. The rest is identical to Case 1.

Case 3. $1/3 < p < 1$

We have

$$r_2 = \frac{-1 - \sqrt{4/p - 3}}{2} < \frac{-1 - \sqrt{4 - 3}}{2} = -1.$$

Therefore, if $C \neq 0$, then by (2.2), we have

$$\limsup_{n \rightarrow \infty} P(n) = \infty,$$

a contradiction since we want $0 \leq P(n) \leq 1$. Therefore, $C = 0$ and

$$P(n) = A + Br_1^n.$$

We wish to show that $A = 0$ by showing

$$\lim_{n \rightarrow \infty} P(n) = 0.$$

Suppose we start with $n \in \mathbb{N}$ and eventually traverse 0. Then the number of times we added 2 is r and the number of times we subtracted 1 is $2r + n$ for some $r \in \mathbb{N}$. Thus we have

$$\begin{aligned} P(n) &\leq \sum_{r=0}^{\infty} \binom{3r+n}{r} p^r (1-p)^{2r+n} = \sum_{r=0}^{\infty} \frac{(3r+n)!}{r!(2r+n)!} p^r (1-p)^{2r+n} \\ &= (1-p)^n \sum_{r=0}^{\infty} \frac{(3r+n)(3r+n-1)\dots(2r+n+1)}{r(r-1)\dots 2 \cdot 1} p^r (1-p)^{2r}. \end{aligned}$$

For $r, n \in \mathbb{N} \cup \{0\}$, let

$$a_{r,n} := \frac{(3r+n)(3r+n-1)\dots(2r+n+1)p^r(1-p)^{2r}}{r(r-1)\dots 2 \cdot 1}.$$

If $n \in \mathbb{N}$ is fixed, then for the sequence $a_{r,n}$, we have

$$a_{r+1,n} \sim a_{r,n} \frac{27p(1-p)^2}{4}.$$

We can work out that $p(1-p)^2 > \frac{4}{27}$ in the range $p \in (1/3, 1)$. Thus for fixed $n \in \mathbb{N} \cup \{0\}$, $a_{r,n}$ is eventually growing slower than a geometric series of common ratio less than 1. Thus

$$\sum_{r=0}^{\infty} a_{r,n} < \infty.$$

Also, for $r, n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} a_{r,n+1} &= \frac{(3r+n+1)(3r+n)\dots(2r+n+2)p^r(1-p)^{2r}}{r(r-1)\dots 2 \cdot 1} \\ &= \frac{(3r+n)(3r+n-1)\dots(2r+n+1)p^r(1-p)^{2r}}{r(r-1)\dots 2 \cdot 1} \cdot \frac{(3r+n+1)}{(2r+n+1)} \\ &< a_{r,n} \cdot \frac{3}{2}. \end{aligned}$$

Thus for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\sum_{r=0}^{\infty} a_{r,n+1} < \frac{3}{2} \sum_{r=0}^{\infty} a_{r,n}.$$

Since $1-p < 2/3$ this gives us

$$\lim_{n \rightarrow \infty} (1-p)^n \sum_{r=0}^{\infty} a_{r,n} = 0.$$

Thus

$$0 \leq \liminf_{n \rightarrow \infty} P(n) \leq \limsup_{n \rightarrow \infty} P(n) \leq \lim_{n \rightarrow \infty} (1-p)^n \sum_{r=0}^{\infty} a_{r,n} = 0$$

so that

$$(2.3) \quad \lim_{n \rightarrow \infty} P(n) = 0.$$

Thus, using (2.3), we have

$$0 = \lim_{n \rightarrow \infty} P(n) = \lim_{n \rightarrow \infty} A + Br_1^n = \lim_{n \rightarrow \infty} A + Br_1^n = A,$$

giving us $A = 0$. Thus

$$P(n) = Br_1^n.$$

Since $P(0) = 1$, we thus have $B = 1$ so

$$P(n) = r_1^n.$$

We recall that $P(2)$ is the probability of starting at 2 and randomly adding 2 or subtracting 1 and eventually traversing 0. Thus $1 - P(2)$ is the probability of never traversing 0, which is equal to the probability of never traversing a pair $(1, 1)$ in the Fibonacci tree after taking the first branch to be a right branch. Since we don't want to rule out the possibility of the first choice being a left branch (from which it is unavoidable to traverse another $(1, 1)$ pair), we therefore have that the

probability of a random path not traversing a pair $(1, 1)$ except at the root is

$$\begin{aligned}
 p(1 - P(2)) &= p(1 - r_1^2) \\
 &= p - p \left(\frac{-1 + \sqrt{4/p - 3}}{2} \right)^2 \\
 &= \frac{4p - p(\sqrt{4/p - 3} - 1)^2}{4} \\
 &= \frac{4p - p(4/p - 2 - 2\sqrt{4/p - 3})}{4} \\
 &= \frac{3p - 2 + p\sqrt{4/p - 3}}{2}.
 \end{aligned}$$

□

Remark 2.1. At $p = 1$, we get that a probability of 1 of choosing the path that only consists of right branches. This path consists of the usual Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$. Therefore, we obtain a probability of 0 of traversing a $(1, 1)$ pair, except at the root.

Example 2.1. For $p = 1/2$ where we do not favour choosing one branch over the other that Theorem 1.1 gives the probability of not traversing a $(1, 1)$ pair after the root is $\frac{\sqrt{5}-1}{4} \approx 0.3090169942$.

3. ASYMPTOTICS FOR $S(n)$ AND $B(n)$

Recall that $B(n)$ counts the number of $(1, 1)$ at level $3n$ such that the path to this level does not traverse a 0, whereas $S(n)$ is definitely similarly except the path does not traverse an intermediate pair $(1, 1)$.

Recall the definitions of $S(n)$ and $B(n)$ given in Definition 1.2. Here we prove that

$$S(n) = \frac{6.75^n}{3\sqrt{3\pi n^{3/2}}} \left(1 + \frac{17}{72n} + O\left(\frac{1}{n^2}\right) \right)$$

and

$$B(n) = \frac{\sqrt{3} \cdot 6.75^n}{4\sqrt{\pi n^{3/2}}} \left(1 - \frac{43}{72n} + O\left(\frac{1}{n^2}\right) \right).$$

Proposition 3.1. We have $S(1) = 5$ and

$$S(n) = \frac{2}{3n-1} \binom{3n-1}{n-1}$$

for $n \geq 2$.

Proof. It is easily seen that $S(1) = 5$, and $S(2) = \frac{2}{5} \binom{5}{1} = 2$. At levels $n \geq 2$, we know that if we traverse a $(1, 1)$ pair at level n , then we must have taken twice as many left branches as right branches. Also, if our first branch is a left branch we will traverse a $(1, 1)$ pair at level 1. Therefore all primitive $(1, 1)$ pairs at level $n \geq 2$ must occur on paths where the initial branch is a right branch. After this initial right branch, we must therefore go through $n-1$ right branches and $2n$ left branches to reach a primitive $(1, 1)$ pair at level n for $n \geq 2$. Therefore for $n \geq 2$, we have $S(n) \leq \binom{3n-1}{n-1}$. This upperbound, however, will over-count the number of primitive $(1, 1)$ s since it also counts paths where the path to an intermediate

pair might have twice as many left branches as right branches when really for any intermediate pair we want the number of left branches to be strictly less than twice the number of right branches. We thus get the recurrence:

$$(3.1) \quad S(n) = \binom{3n-1}{n-1} - \binom{3n-3}{n-1} - \sum_{k=2}^{n-1} \binom{3n-3k}{n-k} S(k)$$

Assuming by induction that $S(1) = 5$ and

$$S(k) = \frac{2}{3k-1} \binom{3k-1}{k-1}$$

for $2 \leq k < n$, one can easily check via Maple that equation (3.1) is satisfied when $S(n) = \frac{2}{3n-1} \binom{3n-1}{n-1}$. \square

Proposition 3.2. *For all $n \in \mathbb{N}$, $n \geq 100$, we have*

$$\frac{\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi n}} \left(1 - \frac{7}{72n} - \frac{1}{1296n^2}\right) < \binom{3n}{n} < \frac{\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi n}} \left(1 - \frac{7}{72n} + \frac{3}{4n^2}\right).$$

Proof. Robbins shows in [7] that, for all $n \in \mathbb{N}$, we have

$$(3.2) \quad \sqrt{2\pi n}^{n+1/2} e^{-n} \cdot e^{1/(12n+1)} < n! < \sqrt{2\pi n}^{n+1/2} e^{-n} \cdot e^{1/(12n)}.$$

Thus we have

$$\binom{3n}{n} = \frac{(3n)!}{(2n)! \cdot n!} < \frac{\sqrt{3} \cdot 6.75^n \cdot e^{-\frac{7}{72n} + \frac{7}{10n^2}}}{2\sqrt{\pi n}}$$

For $-1 < x < 1$, we have

$$e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots$$

Letting $x = \frac{-7}{72n} + \frac{7}{10n^2}$, we have for $n \geq 8$ that $-1 < x < 0$ and hence

$$e^x < 1 + x + \frac{x^2}{2},$$

as it is an alternating series. Thus, by Maple, we have

$$\begin{aligned} \binom{3n}{n} &< \frac{\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi n}} \left(1 - \frac{7}{72n} + \frac{7}{10n^2} + \frac{\left(-\frac{7}{72n} + \frac{7}{10n^2}\right)^2}{2}\right) \\ &< \frac{\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi n}} \left(1 - \frac{7}{72n} + \frac{3}{4n^2}\right). \end{aligned}$$

A similar argument can be used for the opposite inequality. \square

Corollary 3.1. *For all $n \in \mathbb{N}$, $n \geq 100$, we have*

$$\frac{6.75^n}{3\sqrt{3\pi n^{3/2}}} \left(1 + \frac{17}{72n} + \frac{3}{40n^2}\right) < S(n) < \frac{6.75^n}{3\sqrt{3\pi n^{3/2}}} \left(1 + \frac{17}{72n} + \frac{5}{6n^2}\right).$$

and

$$\frac{\sqrt{3} \cdot 6.75^n}{4\sqrt{\pi n^{3/2}}} \left(1 - \frac{43}{72n} + \frac{1}{4n^2}\right) < B(n) < \frac{\sqrt{3} \cdot 6.75^n}{4\sqrt{\pi n^{3/2}}} \left(1 - \frac{43}{72n} + \frac{21}{20n^2}\right).$$

Proof. We can easily deduce our bounds from equation (1.1) and Propositions 3.1 and 3.2. \square

4. THE LIMIT OF THE RATIO

Recall that $A(n)$ is the number of pairs $(1, 1)$ at level $3n$ where there are no restrictions on the path. Here we prove a weak bound for $A(n)$, which we use to derive that

$$\lim_{n \rightarrow \infty} \frac{A(n+1)}{A(n)} = 6.75.$$

Proposition 4.1. *For all $n \in \mathbb{N} \cup \{0\}$, we have*

$$A(n) < 2 \cdot \binom{3n}{n}.$$

Proof. We will prove by induction on n . One can easily check that this holds for $n = 0, 1, \dots, 4$. Suppose for some $n \geq 5$, we have for all $0 \leq i \leq n-1$ that

$$A(i) \leq 2 \binom{3i}{i}.$$

Then by (1.2) we have

$$A(n) < 2 \sum_{i=1}^n \binom{3n-3i}{n-i} S(i).$$

Noticing that for $S(n) = \frac{2}{3n-1} \binom{3n-1}{n-1}$ for $n \geq 2$ and $S(1) = 5 = \frac{2}{3 \cdot 1 - 1} \binom{3 \cdot 1 - 1}{1-1} + 4$, we observe, with the help of maple that

$$\begin{aligned} \frac{2 \sum_{i=1}^n \binom{3n-3i}{n-i} S(i)}{2 \binom{3n}{n}} &= \frac{8 \binom{3n-3}{n-1} + 2 \sum_{i=1}^n \binom{3n-3i}{n-i} \frac{2}{3i-1} \binom{3i-1}{i-1}}{2 \binom{3n}{n}} \\ &= \frac{25n^2 - 17n + 2}{27n^2 - 27n + 6}. \end{aligned}$$

We observe that this is less than 1 for all $n \geq 5$, proving

$$A(n) < 2 \sum_{i=1}^n \binom{3n-3i}{n-i} S(i) < 2 \binom{3n}{n}$$

as desired. □

Corollary 4.1. *We have*

$$A(n) \leq (1 + o(1)) \frac{\sqrt{3} \cdot 6.75^n}{\sqrt{\pi n}}.$$

Proof. This follows from Propositions 3.2 and 4.1. □

Lemma 4.1. *For all $n \in \mathbb{N}$, $n \geq 2$ we have*

$$\frac{S(n+1)}{S(n)} < \frac{S(n+2)}{S(n+1)}.$$

Proof. We show for all $n \geq 2$, we have

$$\frac{S(n+2)S(n)}{(S(n+1))^2} > 1.$$

By Proposition 3.1, we have for each $n \geq 2$

$$\frac{S(n+2)S(n)}{(S(n+1))^2} = \frac{36n^4 + 126n^3 + 158n^2 + 84n + 16}{36n^4 + 126n^3 + 104n^2 - 14n - 12} > 1.$$

□

Lemma 4.2. *For all $n \in \mathbb{N}$, we have*

$$\frac{A(n+1)}{A(n)} < \frac{A(n+2)}{A(n+1)}.$$

Proof. We prove by induction on n . First, for $n = 1$, we have

$$\frac{A(2)}{A(1)} = \frac{27}{5} < \frac{152}{27} = \frac{A(3)}{A(2)}.$$

Suppose by strong induction, we have

$$\frac{A(i+1)}{A(i)} < \frac{A(i+2)}{A(i+1)}$$

for all $1 \leq i \leq n-1$. Then we can deduce that

$$\frac{A(i)}{A(n)} > \frac{A(i+1)}{A(n+1)}$$

for all $1 \leq i \leq n-1$. Also, from (1.2) we have:

$$\begin{aligned} \frac{A(n+1)}{A(n)} &= 5 + \frac{S(n+1)}{S(n)} - \frac{S(n+1)(A(n) - S(n))}{A(n)S(n)} + \frac{A(n+1) - 5 \cdot A(n) - S(n+1)}{A(n)} \\ &= 5 + \frac{S(n+1)}{S(n)} - \frac{S(n+1)}{A(n)S(n)} \left(\sum_{i=1}^{n-1} A(i)S(n-i) \right) + \frac{1}{A(n)} \left(\sum_{i=1}^{n-1} A(i)S(n+1-i) \right) \end{aligned}$$

By Lemma 4.1, we can derive that

$$\frac{S(n+1)}{S(n)} > \frac{S(n+1-i)}{S(n-i)}$$

or

$$\frac{S(n+1)}{S(n)} \cdot S(n-i) - S(n+1-i) > 0$$

for all $1 \leq i < n$. Thus we have the following:

$$\begin{aligned}
\frac{A(n+1)}{A(n)} &= 5 + \frac{S(n+1)}{S(n)} - \sum_{i=1}^{n-1} \frac{A(i)}{A(n)} \left(\frac{S(n+1)}{S(n)} \cdot S(n-i) - S(n+1-i) \right) \\
\frac{A(n+1)}{A(n)} &< 5 + \frac{S(n+1)}{S(n)} - \sum_{i=1}^{n-1} \frac{A(i+1)}{A(n+1)} \left(\frac{S(n+1)}{S(n)} \cdot S(n-i) - S(n+1-i) \right) \\
&= 5 + \frac{S(n+1)}{S(n)} - \frac{S(n+1)}{A(n+1)S(n)} \left(\sum_{i=1}^{n-1} A(i+1)S(n-i) \right) + \frac{1}{A(n+1)} \left(\sum_{i=1}^{n-1} A(i+1)S(n+1-i) \right) \\
&= 5 + \frac{S(n+1)}{S(n)} - \frac{S(n+1)(A(n+1) - 5 \cdot S(n) - S(n+1))}{A(n+1)S(n)} \\
&\quad + \frac{A(n+2) - 5 \cdot A(n+1) - 5 \cdot S(n+1) - S(n+2)}{A(n+1)} \\
&= \frac{S(n+1)^2}{A(n+1)S(n)} + \frac{A(n+2)}{A(n+1)} - \frac{S(n+2)}{A(n+1)} \\
&= \frac{A(n+2)}{A(n+1)} - \frac{S(n+1)}{A(n+1)} \left(\frac{S(n+2)}{S(n+1)} - \frac{S(n+1)}{S(n)} \right) \\
&< \frac{A(n+2)}{A(n+1)}.
\end{aligned}$$

The last inequality follows from Lemma 4.1. Thus, by strong induction, we have our result. \square

Proposition 4.2. *We have*

$$\lim_{n \rightarrow \infty} \frac{A(n+1)}{A(n)} = 6.75.$$

Proof. By Lemma 4.2, we have $\frac{A(n+1)}{A(n)}$ is an increasing sequence in $n \in \mathbb{N} \cup \{0\}$ so the desired limit exists. First, suppose that

$$\lim_{n \rightarrow \infty} \frac{A(n+1)}{A(n)} > 6.75.$$

Then

$$\lim_{n \rightarrow \infty} \frac{A(n)}{6.75^n} = \infty.$$

But by Corollary 4.1, we have

$$\lim_{n \rightarrow \infty} \frac{A(n)}{6.75^n} = 0,$$

a contradiction.

Now suppose

$$\lim_{n \rightarrow \infty} \frac{A(n+1)}{A(n)} < 6.75.$$

Then there exists $r < 6.75$ such that

$$\lim_{n \rightarrow \infty} \frac{A(n)}{r^n} = 0$$

and so

$$\lim_{n \rightarrow \infty} \frac{S(n)}{r^n} = 0.$$

But by Corollary 3.1, we have

$$\lim_{n \rightarrow \infty} \frac{S(n)}{r^n} = \infty,$$

again a contradiction. Thus the result follows. \square

5. PRELIMINARY RESULTS CONCERNING OTHER COPRIME PAIRS

In this section, we turn our attention to the behaviour of other coprime pairs other than $(1, 1)$ and establish a number of useful preliminary results concerning them.

Rittaud constructed a subtree \mathbf{R} from the Fibonacci tree consisting of all the shortest paths from the root $(1, 1)$ down to each coprime pair (a, b) , calling it the restricted tree. The top part of of this subtree is shown in Figure 2. He proves the

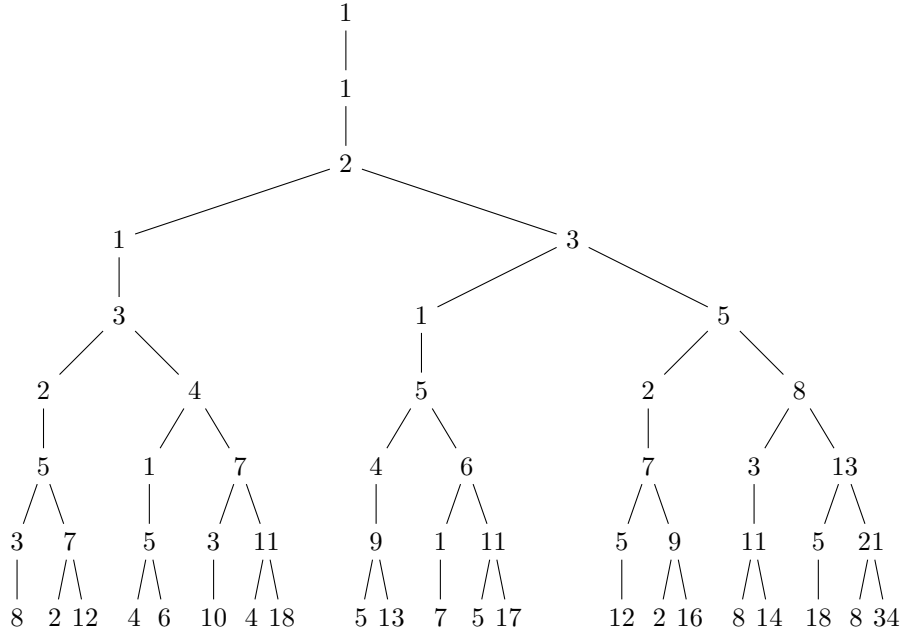


FIGURE 2. The Restricted Tree $\mathbf{R} = \mathbf{R}_{(1,1)}$

following in [6]:

Lemma 5.1 (Rittaud). *The restricted tree \mathbf{R} consists of all paths that do not have two left branches occurring with no right branch between them. Therefore, for all coprime pairs (a, b) , the shortest path from the root $(1, 1)$ to (a, b) does not have two left branches occurring with no right branch between them.*

Lemma 5.2. *Take a coprime pair (a, b) and suppose that the shortest path in the Fibonacci tree from the root $(1, 1)$ to the pair (a, b) consists of k branches. Then for all $n \in \mathbb{N} \cup \{0\}$, there exists $k + 3n$ branches down the tree a pair (a, b) .*

Proof. We know it holds for $n = 0$ for this gives the shortest path. Suppose by induction it holds for some $n \in \mathbb{N} \cup \{0\}$. Then from the ending pair (a, b) of a path of $k + 3n$ we can take a right branch and then two left branches to give the following addition three terms:

$$a + b, a, b.$$

Adding these three branches on to our original path gives a path to the pair (a, b) consisting of $k + 3(n + 1)$. By induction, the result follows. \square

Proposition 5.1. *Take a coprime pair (a, b) that isn't $(1, 1)$ and a path down the Fibonacci tree from the root $(1, 1)$ to a pair (a, b) . Then this path must contain a pair (a, b) where the parent of that specific a is $|a - b|$ in the path.*

Note that the terminal pair (a, b) may not be the only occurrence of the pair (a, b) that the path traverses.

Proof. Suppose the length of this shortest path is k branches. Recall that if you go $3n$ branches down the tree, you arrive at coprime pairs (a, b) where both a and b are odd. If you go $3n + 1$ branches down the tree, you arrive at coprime pairs where a is odd and b is even. Finally, if you go $3n + 2$ branches down the tree, you arrive at coprime pairs where a is even and b is odd. Then by Lemma 5.2, all of the possible number of branches in a path from $(1, 1)$ to (a, b) is $k + 3n$ where $n \in \mathbb{N} \cup \{0\}$. We will prove by induction on n .

For $n = 0$, we obtain the shortest path from the root $(1, 1)$ to (a, b) . By [6, Corollary 5.1] we have that the parent of a of the ending pair (a, b) is $|a - b|$.

Suppose now the proposition holds for all $0 \leq n < N$ for some $N \in \mathbb{N}$. Take a path from $(1, 1)$ to (a, b) consisting of $k + 3N$ branches. If the first branch is a left branch, then we will traverse another $(1, 1)$ pair 3 branches down the tree and so we can remove these first three branches to obtain a path of length $k + 3(N - 1)$ from which by induction the path must consist of a pair (a, b) such that the parent of this specific a is $|a - b|$. Since we only removed the first three branches of the original path, the original path must have this property too.

Suppose that the path in question starts with a right branch. We know that this path isn't the shortest path since $N \geq 1$. Therefore, by Lemma 5.2, we must have that the path consists of two left branches with no right branches between them. Since the first branch is a right branch, it therefore follows that somewhere in the tree we have a consecutive sequence of 3 branches consisting of a right branch followed by two left branches. Suppose the branch immediately before this right branch (in case this specific right branch is the first branch in the path consider the root $(1, 1)$ here) consists of the pair (c, d) . Then taking the right branch and then the two left branches gives us the sequence (starting with the (c, d) pair)

$$c, d, c + d, c, d$$

Therefore the second left branch also consists of the pair (c, d) . Removing the right branch and the two left branches therefore gives us a shorter path to the pair (a, b) . Since by induction this shorter path must have a pair (a, b) with this specific a having a parent of $|a - b|$ in the path, we therefore obtain that the original path has this property too. By induction we obtain our result. \square

Recall the definition of $T_{(a,b)}$ given in the first section.

Lemma 5.3. *Take the tree $\mathbf{T}_{(a,b)}$ for some coprime pair (a,b) . Suppose we take a finite path down the tree consisting of exactly twice as many left branches as right branches, but such that at any given intermediate point the number of left branches taken is less than or equal to twice the number of right branches taken. Then the pair on the last branch will be (a,b) .*

Proof. We prove by induction on n where the n is the number of right branches in the path. First, for $n = 0$, we have a path consisting of no branches and so we just end at the root (a,b) . Suppose by induction the proposition holds for some $n \in \mathbb{N} \cup \{0\}$ and suppose we have a path starting at the root (a,b) with there being $3n$ branches in the path with $n+1$ of them being right branches and $2n+2$ of them being left branches so that at any intermediate point the number of left branches taken is less than or equal to twice the number of right branches taken. Since the first branch will therefore have to be a right branch and the number of left branches in total is exactly twice the number of right branches in total, it follows that somewhere in the path there must be a right branch immediately following by a left branch, which is again immediately followed by another left branch. As in the proof of the last proposition, such a configuration of three branches can be dropped out without affecting the pairing on the last branch in the path. Therefore, removing them will result in a path that will satisfy the proposition with n right branches and so by induction, the pairing on the last branch will be (a,b) . Therefore the pairing on the last branch of the original path must also be (a,b) . By induction, we have our result. \square

Lemma 5.4. *Take a path in $\mathbf{T}_{(a,b)}$ that starts at the root (a,b) . Suppose that the number of left branches is less than twice the number of right branches in this path. Also suppose that at any given intermediate point in the path the number of left branches taken is less than or equal to twice the number of right branches taken. Then the pair on the final branch will not be (a,b) .*

Proof. Suppose for a contradiction that we have such a path with the final pair being (a,b) . We can extend this path by a series of left branches until we get a path having exactly twice as many left branches as right branches and satisfying Lemma 5.3. By Lemma 5.3, the ending pair will be (a,b) . Thus by taking a series of left branches from the pair (a,b) , we get back to the pair (a,b) . Note that the left child of the pair (a,b) is $|a-b| < \max\{a,b\}$. Hence by induction, any term in the left most path from (a,b) will be strictly less than $\max\{a,b\}$ and hence the pair (a,b) cannot repeat. \square

Lemma 5.5. *Take the tree $\mathbf{T}_{(a,b)}$ for some coprime pair (a,b) and let $n \in \mathbb{N}$. Suppose we take all (a,b) pairs that are $3n$ branches down the tree such that the paths to these (a,b) pairs satisfies the following. Let the first branch be a right branch and the branch after any intermediate pair (a,b) in the path be a right branch. The number of such (a,b) pairs is $B(n)$.*

Proof. We will first show that the paths in question are characterised by having twice as many left branches as right branches such that at any given intermediate point the number of left branches taken is less than or equal to twice the number of right branches taken. First off, a path characterised as such will begin with a right branch and at the first point, whether it be some intermediate point or at the final branch, the number of left branches will stop being less than twice the number

of right branches and will instead be equal to it. By Lemma 5.4, the pairing we encounter at this branch will be (a, b) . If this is an intermediate point, then we must take a right branch to preserve the inequality. This will continue on until we come to the last branch that will also have the pair (a, b) . Thus such a path satisfies the paths as described in this lemma.

Conversely, a path described as in this lemma begins with a right branch and when it traverses an (a, b) pair again, we must have twice as many left branches as right branches by Lemmas 5.3 and 5.4. Then we take another right branch and so on. This fits the characterisation we have given. Thus it has become a question of counting the number of paths that are characterised as in the start of the proof. By using the definitions of $S(n)$ and $B(n)$ and Proposition 2.1, we can see that this is $B(n)$. \square

Lemma 5.6. *Take the tree $\mathbf{T}_{(a,b)}$ for some coprime pair (a, b) . Then for all paths to another (a, b) further down the tree such that the first branch is a left branch, there must exist a pair (a, b) in the path such that the parent of that specific a is $|a - b|$ in the path.*

Proof. We prove our result on n where $3n$ is the length of the path in question. For $n = 1$, we have the sequence

$$a, b, |a - b|, a, b.$$

Suppose it holds for some $n \in \mathbb{N}$. We want to show it holds for $n + 1$. So consider a path of length $3n + 3$ that starts at the root (a, b) where the first branch is a left branch and ends at a pair (a, b) . Without loss of generality, we may assume that it doesn't traverse any intermediate pair (a, b) before the final (a, b) . Thus we wish to show that the third last term in the sequence is $|a - b|$. Suppose for a contradiction it doesn't. Then the third last term must be $a + b$. Since $b = |(a + b) - b|$, the final branch must be a left branch. Also since $a < a + b$, the second last branch must also be a left branch. Thus somewhere in the path there must be a right branch immediately followed by a left branch, which is again immediately followed by another left branch. As in the proofs of Proposition 5.1 and Lemma 5.3 such a configuration can be dropped out without affecting the pairing on the last branch (a, b) . But then this smaller path would not have any intermediate (a, b) pairs and the third last term would still be $a + b$, which isn't possible by our inductive assumption. Therefore, the third last term of the original path had to have been $|a - b|$ as well. Thus we have our result. \square

Corollary 5.1. *Take the tree $\mathbf{T}_{(|a-b|,a)}$ for some coprime pair (a, b) . Then for all paths to an (a, b) , there must exist a pair (a, b) in the path such that the parent of that specific a is $|a - b|$ in the path.*

Proof. Take such a path to a pair (a, b) and suppose there exists no pair (a, b) in that path such that the parent of that specific a in the path is $|a - b|$. Adding on a and b at the front will give a path that contradicts Lemma 5.6. Therefore the result follows. \square

Proposition 5.2. *Take the tree $\mathbf{T}_{(a,b)}$ for some coprime pair (a, b) . Then for all $n \in \mathbb{N}$ the number of (a, b) pairs that are $3n$ branches down the tree such that there is no $(|a - b|, a)$ pair in any of the paths is $B(n)$.*

Proof. Combine Lemmas 5.5 and 5.6 and Corollary 5.1. \square

Proposition 5.3. *Take a coprime pair (a, b) that is not the $(1, 1)$ pair and suppose the shortest walk from the root $(1, 1)$ to (a, b) consists of k branches. Then we have for all $n \geq \lfloor \frac{k}{3} \rfloor$*

$$A_{(a,b)}(n) = \sum_{i=\lfloor \frac{k-1}{3} \rfloor}^n A_{(|a-b|,a)}(i)B(n-i)$$

if either a is even or b is even, and

$$A_{(a,b)}(n) = \sum_{i=\lfloor \frac{k-1}{3} \rfloor}^{n-1} A_{(|a-b|,a)}(i)B(n-1-i)$$

if a and b are both odd.

Proof. By Proposition 5.1 any path in the Fibonacci tree that starts at the root $(1, 1)$ and ends at the pair (a, b) at level n must contain the pair $(|a-b|, a)$. Consider the last place in a given path that this pair occurs and say it occurs at level i . Then by Corollary 5.1 the next element in the path is b and, by Proposition 5.2, this gives rise to $B(n-i)$ pairs of (a, b) at level n or $n+1$ (depending on the parity of a and b). Conversely, every pair $(|a-b|, a)$ that occurs at an intermediate level i gives rise to $B(n-i)$ paths to pairs of (a, b) at level n or $n+1$. The summation starts at $i = \lfloor \frac{k-1}{3} \rfloor$ since the shortest path to $(|a-b|, a)$ consists of $k-1$ branches and so occurs at level $\lfloor \frac{k-1}{3} \rfloor$. Thus the formula follows. \square

Corollary 5.2. *For all $n \in \mathbb{N} \cup \{0\}$, we have*

$$A_{(1,2)}(n) = \frac{A_{(1,1)}(n+1) - B(n+1)}{4}.$$

Proof. By Proposition 5.3, we have for all $n \in \mathbb{N} \cup \{0\}$

$$(5.1) \quad A_{(1,2)}(n) = \sum_{i=0}^n A_{(1,1)}(i)B(n-i).$$

All paths down to a $(1, 1)$ pair at level n must satisfy exactly one of the following two conditions. Either for all other $(1, 1)$ pairs it traverses it takes a right branch immediately afterwards, or there exists a first $(1, 1)$ pair where the path takes a left branch immediately afterwards, consequently ending up with a choice of 4 $(1, 1)$ pairs at the very next level. Thus for all $n \in \mathbb{N} \cup \{0\}$ we have

$$A_{(1,1)}(n) = B(n) + \sum_{i=0}^{n-1} B(i) \cdot 4 \cdot A_{(1,1)}(n-1-i).$$

Relabeling the index in the summation gives

$$(5.2) \quad A_{(1,1)}(n) = B(n) + 4 \sum_{i=0}^{n-1} A_{(1,1)}(i)B(n-1-i).$$

Substituting in (5.1) we have for all $n \in \mathbb{N}$

$$A_{(1,1)}(n) = B(n) + 4 \cdot A_{(1,2)}(n-1).$$

Thus we have our result. \square

Corollary 5.3. *Take two pairs of coprime positive integers (a, b) and (c, d) and suppose that the shortest paths from the root $(1, 1)$ to (a, b) and (c, d) consist of the same number of branches. Then we have*

$$A_{(a,b)}(n) = A_{(c,d)}(n)$$

for all $n \in \mathbb{N} \cup \{0\}$.

Proof. This is an easy induction on the number of branches in the shortest paths, using the result of Proposition 5.3. \square

Corollary 5.4. *For all $n \in \mathbb{N} \cup \{0\}$, we have*

$$A_{(2,1)}(n) = A_{(2,3)}(n) = A_{(1,1)}(n+1) - 4 \cdot A_{(1,1)}(n).$$

Proof. By Corollary 5.3 it suffices to prove that

$$A_{(2,1)}(n) = A_{(1,1)}(n+1) - 4 \cdot A_{(1,1)}(n).$$

since the pairs $(2, 1)$ and $(2, 3)$ have shortest paths consisting of 2 branches each. First, consider all $(2, 1)$ pairs at level n . If we take an immediate left branch we encounter $(1, 1)$ pairs at level $n+1$. Now consider all $(1, 1)$ pairs at level n . The paths to these $(1, 1)$ pairs must either have the element 0 or the element 2 immediately before the $(1, 1)$ pair. There are $4 \cdot A(n)$ pairs $(1, 1)$ of the former type since following backwards along the path will give us 4 $(1, 1)$ pairs at level n . Therefore the number of $(1, 1)$ pairs with a path that has the element 2 immediately before the $(1, 1)$ pair is $A(n+1) - 4 \cdot A(n)$. Since the second and third last elements of these paths form $(2, 1)$ pairs we have, by our observation that all $(2, 1)$ pairs have a $(1, 1)$ immediately beneath them, our result. \square

Lemma 5.7. *Take a coprime pair (a, b) that is not the $(1, 1)$ pair and suppose the shortest walk from the root $(1, 1)$ to (a, b) consists of $k \geq 3$ branches. Suppose the last five numbers in the corresponding sequence of the shortest path, including the last two numbers a and b are*

$$a_0, a_1, a_2, a, b.$$

For all $n \geq \lfloor \frac{k}{3} \rfloor$, we have

$$A_{(a,b)}(n) = A_{(a_1,a_2)}(n) - A_{(a_0,a_1)}(n)$$

if a is odd, and we have

$$A_{(a,b)}(n) = A_{(a_1,a_2)}(n+1) - A_{(a_0,a_1)}(n).$$

if a is even (and hence b is odd).

Proof. We prove by induction on the number of branches in the shortest path from the root $(1, 1)$ to the pair (a, b) . First, suppose (a, b) is a pair with a shortest path of 3 branches. Then both a and b are odd. By Proposition 5.3, we have

$$A_{(a,b)}(n) = \sum_{i=0}^{n-1} A_{(|a-b|,a)}(i) B(n-1-i)$$

for all $n \in \mathbb{N}$ where $(|a-b|, a)$ is a pair with a shortest path of 2 branches. There are only two pairs that $(|a-b|, a)$ can be: $(2, 1)$ or $(2, 3)$. Thus by Corollary 5.4,

we have

$$\begin{aligned} A_{(a,b)}(n) &= \sum_{i=0}^{n-1} (A_{(1,1)}(i+1) - 4 \cdot A_{(1,1)}(i)) B(n-1-i) \\ &= \sum_{i=0}^{n-1} A_{(1,1)}(i+1) B(n-1-i) - 4 \sum_{i=0}^{n-1} A_{(1,1)}(i) B(n-1-i) \end{aligned}$$

By Corollary 5.3 and (5.2), we have

$$\begin{aligned} A_{(a,b)}(n) &= A_{(a,b)}(n) - B(n) - 4 \sum_{i=0}^{n-1} A_{(1,1)}(i) B(n-1-i) \\ &= A_{(1,2)}(n) - A_{(1,1)}(n). \end{aligned}$$

Thus it holds for all $n \in \mathbb{N}$ for the pair (a, b) since $(1, 2)$ has a shortest path having only 1 branch and $(1, 1)$ has a shortest path of 0 branches. Suppose the proposition holds for pairs that have a shortest path of length $k-1$ branches for some $k \geq 4$ and suppose we want to show it holds for pairs with shortest paths of lengths k . Let (a, b) be a pair with a shortest path of length k . Let the last six elements of the shortest path be to (a, b) to be

$$a_0, a_1, a_2, |a-b|, a, b.$$

First, suppose that both a and b are odd. Then, by Proposition 5.3, we have

$$A_{(a,b)}(n) = \sum_{i=\lfloor \frac{k-1}{3} \rfloor}^{n-1} A_{(|a-b|,a)}(i) B(n-1-i)$$

for all $n \geq \frac{k}{3}$. By our inductive hypothesis, we have

$$A_{(|a-b|,a)}(i) = A_{(a_1,a_2)}(i+1) - A_{(a_0,a_1)}(i)$$

for all $i \geq \lfloor \frac{k-1}{3} \rfloor$ since $|a-b|$ is even and a is odd. Then we have

$$\begin{aligned} A_{(a,b)}(n) &= \sum_{i=\lfloor \frac{k-1}{3} \rfloor}^{n-1} (A_{(a_1,a_2)}(i+1) - A_{(a_0,a_1)}(i)) B(n-1-i) \\ &= \sum_{i=\lfloor \frac{k-1}{3} \rfloor}^{n-1} A_{(a_1,a_2)}(i+1) B(n-1-i) - \sum_{i=\lfloor \frac{k-1}{3} \rfloor}^{n-1} A_{(a_0,a_1)}(i) B(n-1-i) \\ &= \sum_{i=\lfloor \frac{k-1}{3} \rfloor+1}^n A_{(a_1,a_2)}(i) B(n-i) - \sum_{i=\lfloor \frac{k-1}{3} \rfloor}^{n-1} A_{(a_0,a_1)}(i) B(n-1-i) \\ &= \sum_{i=\lfloor \frac{k-3}{3} \rfloor+1}^n A_{(a_1,a_2)}(i) B(n-i) - \sum_{i=\lfloor \frac{k-4}{3} \rfloor+1}^{n-1} A_{(a_0,a_1)}(i) B(n-1-i) \end{aligned}$$

Thus, by Proposition 5.3, we have

$$A_{(a,b)}(n) = A_{(a_2,|a-b|)}(n) - B\left(n - \left\lfloor \frac{k-3}{3} \right\rfloor\right) - \left(A_{(a_1,a_2)}(n) - B\left(n-1 - \left\lfloor \frac{k-4}{3} \right\rfloor\right)\right)$$

since the shortest path to $(a_2, |a - b|)$ consists of $k - 2$ branches and the shortest path to (a_1, a_2) consists of $k - 3$ branches. Also, we have $\lfloor \frac{k-3}{3} \rfloor = 1 + \lfloor \frac{k-4}{3} \rfloor$ since $3|k$. Thus we get our result

$$A_{(a,b)}(n) = A_{(a_2, |a-b|)}(n) - A_{(a_1, a_2)}(n)$$

for all $n \geq \frac{k}{3}$.

By a similar argument, if a is odd and b is even, then

$$A_{(a,b)}(n) = A_{(a_2, |a-b|)}(n) - A_{(a_1, a_2)}(n).$$

Also, by a similar argument, if a is even and b is odd, then

$$A_{(a,b)}(n) = A_{(a_2, |a-b|)}(n+1) - A_{(a_1, a_2)}(n).$$

□

6. ASYMPTOTICS FOR $A(n)$

We establish our asymptotic results concerning $A(n)$ here. First, we prove the following lemma: Note that $A_{1,1}(n) = A(n)$.

Lemma 6.1. *For all $n \in \mathbb{N} \cup \{0\}$, we have*

$$A(n+2) - 16 \cdot A(n+1) + 64 \cdot A(n) = B(n+2) + 4 \cdot S(n+2).$$

Proof. By Proposition 5.3, we have, for all $n \in \mathbb{N} \cup \{0\}$

$$A_{(2,1)}(n) = \sum_{i=0}^n A_{(1,2)}(i) B(n-i).$$

By Corollaries 5.2 and 5.4, we therefore have

$$A(n+1) - 4 \cdot A(n) = \sum_{i=0}^n \frac{A(i+1) - B(i+1)}{4} B(n-i).$$

Thus we have the following:

$$A(n+1) - 4 \cdot A(n) = \frac{1}{4} \sum_{i=1}^{n+1} A(i) B(n+1-i) - \frac{1}{4} \sum_{i=0}^n B(i+1) B(n-i).$$

By (5.2), we thus have

$$\begin{aligned} A(n+1) - 4 \cdot A(n) &= \frac{1}{4} \left(\frac{A(n+2) - B(n+2)}{4} - B(n+1) \right) - \frac{1}{4} \sum_{i=0}^n B(i+1) B(n-i) \\ (6.1) \quad &= \frac{A(n+2) - B(n+2)}{16} - \frac{B(n+1)}{4} - \frac{1}{4} \sum_{i=0}^n B(i+1) B(n-i). \end{aligned}$$

In [2] we have the combinatorial identity

$$\sum_{k=0}^n \binom{tk+r}{k} \binom{tn-tk+s}{n-k} \frac{r}{tk+r} \cdot \frac{s}{tn-tk+s} = \binom{tn+r+s}{n} \frac{r+s}{tn+r+s}$$

valid for all $n \in \mathbb{N}$ and all $r, s, t \in \mathbb{R}$. Substituting in $t = 3$, $r = 1$, and $s = 1$ gives us

$$(6.2) \quad \sum_{k=0}^n \binom{3k+1}{k} \binom{3n-3k+1}{n-k} \frac{1}{3k+1} \cdot \frac{1}{3n-3k+1} = \binom{3n+2}{n} \frac{2}{3n+2}.$$

Also for all $n \in \mathbb{N} \cup \{0\}$, we have

$$B(n) = \frac{1}{2n+1} \binom{3n}{n} = \frac{1}{2n+1} \frac{(3n)!}{n!(2n)!} = \frac{(3n)!}{n!(2n+1)!} = \frac{1}{3n+1} \binom{3n+1}{n}.$$

Thus by (6.2) and Proposition 3.1 we have for all $n \in \mathbb{N}$

$$\sum_{k=0}^n B(k)B(n-k) = S(n+1).$$

Thus for all $n \in \mathbb{N}$ we have

$$\sum_{i=0}^n B(i+1)B(n-i) = \sum_{i=1}^{n+1} B(i)B(n+1-i) = S(n+2) - B(n+1).$$

Thus by (6.1) we have for all $n \in \mathbb{N}$

$$\begin{aligned} A(n+1) - 4 \cdot A(n) &= \frac{A(n+2) - B(n+2)}{16} - \frac{B(n+1)}{4} - \frac{1}{4}(S(n+2) - B(n+1)) \\ &= \frac{A(n+2) - B(n+2)}{16} - \frac{S(n+2)}{4}. \end{aligned}$$

Thus, for all $n \in \mathbb{N}$, we have our result. \square

Proposition 6.1. *We have*

$$A(n) = \frac{243 \cdot 6.75^n}{4\sqrt{3\pi}n^{3/2}}(1 + o(1)).$$

Proof. We can derive this from Propositions 3.1 and 4.2, Corollary 3.1, and Lemma 6.1. \square

Proposition 6.2. *For all $n \in \mathbb{N}$, $n \geq 100$, we have*

$$\frac{25 \cdot 6.75^{n+2}}{12\sqrt{3\pi}(n+2)^{3/2}} \left(1 - \frac{23}{360(n+2)} + \frac{69}{500(n+2)^2}\right) < B(n+2) + 4 \cdot S(n+2)$$

and

$$B(n+2) + 4 \cdot S(n+2) < \frac{25 \cdot 6.75^{n+2}}{12\sqrt{3\pi}(n+2)^{3/2}} \left(1 - \frac{23}{360(n+2)} + \frac{23}{25(n+2)^2}\right).$$

Proof. We can get our bounds from Corollary 3.1. \square

Definition 6.1. *Define $D : \mathbb{N} \cup \{0\} \rightarrow \mathbb{Z}$ as $D(0) = -3$ and*

$$D(n+1) = 8 \cdot D(n) + B(n+2) + 4 \cdot S(n+2).$$

for all $n \in \mathbb{N} \cup \{0\}$. It can easily be verified with the help of Lemma 6.1 that, for all $n \in \mathbb{N} \cup \{0\}$, we have

$$(6.3) \quad D(n) = A(n+1) - 8 \cdot A(n).$$

Lemma 6.2. *We have*

$$D(n) \sim \frac{-405\sqrt{3} \cdot 6.75^n}{16\sqrt{\pi}n^{3/2}}.$$

Proof. The lemma can easily be verified with (6.3) and Proposition 6.1. \square

Proposition 6.3. *For all $n \geq 100$, we have*

$$\left(-\frac{405\sqrt{3} \cdot 6.75^n}{16\sqrt{\pi}n^{3/2}}\right) \left(1 - \frac{4019}{360n} + \frac{16072}{45n^2}\right) < D(n) < \left(-\frac{405\sqrt{3} \cdot 6.75^n}{16\sqrt{\pi}n^{3/2}}\right) \left(1 - \frac{4019}{360n}\right).$$

Proof. We first prove that

$$D(n) < \left(-\frac{405\sqrt{3} \cdot 6.75^n}{16\sqrt{\pi}n^{3/2}}\right) \left(1 - \frac{4019}{360n}\right)$$

for all $n \geq 100$. Suppose for a contradiction that for some $n \geq 100$, we have

$$D(n) \geq \left(-\frac{405\sqrt{3} \cdot 6.75^n}{16\sqrt{\pi}n^{3/2}}\right) \left(1 - \frac{4019}{360n}\right).$$

The right-hand side of the above inequality is a transcendental number for all $n \in \mathbb{N}$, and since $D(n) \in \mathbb{Z}$ for all $n \in \mathbb{N}$, we must therefore have that

$$D(n) > \left(-\frac{405\sqrt{3} \cdot 6.75^n}{16\sqrt{\pi}n^{3/2}}\right) \left(1 - \frac{4019}{360n}\right).$$

Thus we have

$$D(n) = \left(-\frac{405\sqrt{3} \cdot 6.75^n}{16\sqrt{\pi}n^{3/2}}\right) \left(1 - \frac{C}{n}\right).$$

where $C > \frac{4019}{360}$. Then, by Proposition 6.2, we have

$$\begin{aligned} D(n+1) &= 8 \cdot D(n) + B(n+2) + 4 \cdot S(n+2) \\ &> \left(-\frac{405\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi}(n+1)^{3/2}}\right) \left(\frac{n+1}{n}\right)^{3/2} \left(1 - \frac{C}{n}\right) \\ &\quad + \frac{25 \cdot 6.75^{n+2}}{12\sqrt{3}\pi(n+1)^{3/2}} \left(\frac{n+1}{n+2}\right)^{3/2} \left(1 - \frac{23}{360(n+2)}\right). \end{aligned}$$

By obtaining good enough bounds for $\left(\frac{n+1}{n}\right)^{3/2}$ and $\left(\frac{n+1}{n+2}\right)^{3/2}$ using the binomial theorem expansion of $(1+x)^{-3/2}$, we can deduce that following bound (see Appendix):

$$D(n+1) > \left(-\frac{405\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi}(n+1)^{3/2}}\right) \left(1 + \frac{3-2C}{2(n+1)}\right) + \left(\frac{25 \cdot 6.75^{n+2}}{12\sqrt{3}\pi(n+1)^{3/2}}\right) \left(1 - \frac{563}{360(n+1)}\right).$$

Thus we have

$$D(n+1) > \frac{-405\sqrt{3} \cdot 6.75^{n+1}}{16\sqrt{\pi}(n+1)^{3/2}} \left(1 - \frac{2304C - 4019}{1944(n+1)}\right).$$

We deduce (see Appendix)

$$r := \frac{2304C - 4019}{1944C} > 1.$$

Repeating the above argument as many times as necessary (see Appendix), we deduce, for all $k \in \mathbb{N}$,

$$D(n+k) = \left(-\frac{405\sqrt{3} \cdot 6.75^{n+k}}{16\sqrt{\pi}(n+k)^{3/2}}\right) \left(1 - \frac{C_k}{(n+k)}\right)$$

where $C_k > r^k C$. This leads to

$$\lim_{k \rightarrow \infty} \frac{r^k}{k} = 0,$$

which doesn't hold since $r > 1$, a contradiction. Thus we have our first desired inequality for all $n \geq 100$.

We now prove that

$$D(n) > \left(-\frac{405\sqrt{3} \cdot 6.75^n}{16\sqrt{\pi}n^{3/2}} \right) \left(1 - \frac{4019}{360n} + \frac{16072}{45n^2} \right)$$

for all $n \geq 100$. Suppose for a contradiction that there exists $n \geq 100$ such that

$$D(n) = \left(-\frac{405\sqrt{3} \cdot 6.75^n}{16\sqrt{\pi}n^{3/2}} \right) \left(1 - \frac{4019}{360n} + \frac{C}{n} \right).$$

where $\frac{16072}{45n} \leq C$. Applying the same techniques as in the first inequality (see Appendix), we can obtain

$$D(n+1) < \left(-\frac{405\sqrt{3} \cdot 6.75^{n+1}}{16\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 - \frac{4019}{360(n+1)} + \frac{10C}{9(n+1)} \right).$$

We can again repeat the argument as many times as necessary (see Appendix) to get that, for all $k \in \mathbb{N}$,

$$D(n+k) < \left(-\frac{405\sqrt{3} \cdot 6.75^{n+k}}{16\sqrt{\pi}(n+k)^{3/2}} \right) \left(1 - \frac{4019}{360(n+k)} + \frac{\left(\frac{10}{9}\right)^k C}{(n+k)} \right).$$

This leads to

$$\lim_{k \rightarrow \infty} \frac{\left(\frac{10}{9}\right)^k}{k} = 0,$$

which doesn't hold, a contradiction. Thus we have our second desired inequality for all $n \geq 100$. \square

Proof of Theorem 1.2. We can verify the inequalities for all $n < 100$ with Maple. Thus we need only prove the inequalities hold for all $n \geq 100$. The proof of these inequalities follows the same procedure as in the proof of Proposition 6.3 with $D(n)$ replaced by $A(n)$ and $B(n+2) + 4 \cdot S(n+2)$ replaced by $D(n)$, using (6.3) and Proposition 6.3. \square

Remark 6.1. *Theorem 1.2 provides very good estimates for $A(n)$ for all $n \in \mathbb{N}$. Define $(C_n)_{n \in \mathbb{N}}$ by*

$$A(n) = \frac{C_n \cdot 6.75^n}{n^{3/2}}.$$

Then Theorem 1.2 gives us

$$C \left(1 - \frac{1387}{72n} \right) < C_n < C \left(1 - \frac{1387}{72n} + \frac{5548}{9n^2} \right)$$

where

$$C = \frac{243}{4\sqrt{3\pi}} = 19.78840173\dots,$$

leading to an error for C_n to be

$$\frac{243}{4\sqrt{3\pi}} \cdot \frac{5548}{9n^2} = \frac{37449}{\sqrt{3\pi}n^2} < \frac{12199}{n^2}.$$

For example, we have

$$19.407 < C_{1000} < 19.42$$

and

$$19.7502 < C_{10000} < 19.7505.$$

7. TIGHT BOUNDS FOR $A_{(a,b)}(n)$

Finally, we establish our asymptotic results for other coprime pairs $A_{(a,b)}(n)$ for all coprime ordered pairs (a, b) . First, from Theorem 1.2 and using results from Section 4, we can easily derive the asymptotic formulas for the pairs $(1, 2)$, $(2, 1)$, and $(2, 3)$.

Proposition 7.1. *For all $n \in \mathbb{N}$, we have*

$$\frac{405 \cdot 6.75^n}{4\sqrt{3\pi}n^{3/2}} \left(1 - \frac{60877}{2880n} + \frac{29}{n^2}\right) < A_{(1,2)}(n) < \frac{405 \cdot 6.75^n}{4\sqrt{3\pi}n^{3/2}} \left(1 - \frac{60877}{2880n} + \frac{669}{n^2}\right).$$

Proof. From Corollaries 3.1 and 5.2 and Theorem 1.2, we can deduce the desired bounds. \square

Proposition 7.2. *For all $n \in \mathbb{N}$, we have*

$$\frac{2673 \cdot 6.75^n}{16\sqrt{3\pi}n^{3/2}} \left(1 - \frac{18173}{792n} - \frac{16072}{99n^2}\right) < A_{(2,1)}(n) = A_{(2,3)}(n) < \frac{2673 \cdot 6.75^n}{16\sqrt{3\pi}n^{3/2}} \left(1 - \frac{18173}{792n} + \frac{88768}{99n^2}\right).$$

Proof. By Corollary 5.4, we have

$$A_{(2,1)} = A_{(2,3)} = D(n) + 4 \cdot A_{(1,1)}(n).$$

Applying Proposition 6.3 and Theorem 1.2 gives us the desired bounds. \square

We are now ready to prove our main result concerning the asymptotic formulas for all coprime pairs (a, b) .

Proof of Theorem 1.3. First, we claim that the constants $C_{(a,b)}$ in the Theorem have the form

$$C_{(a,b)} = \frac{243t_k}{4\sqrt{3\pi}}$$

if the shortest path from the root $(1, 1)$ to (a, b) consists of k branches where for all $k \in \mathbb{N} \cup 0$ we have

$$t_{3k} = \left(\frac{1}{2}\right)^k \left(1 + \frac{k}{3}\right),$$

$$t_{3k+1} = \left(\frac{1}{2}\right)^k \left(\frac{5}{3} + \frac{k}{2}\right),$$

and

$$t_{3k+2} = \left(\frac{1}{2}\right)^k \left(\frac{11}{4} + \frac{3k}{4}\right).$$

Note that $t_0 = 1$, $t_1 = \frac{5}{3}$, and $t_2 = \frac{11}{4}$. It's easy to verify that, for all $k \geq 3$, t_k satisfies the recurrence

$$t_k = t_{k-2} - t_{k-3}$$

if $3 \nmid k+1$ and

$$t_k = 6.75 \cdot t_{k-2} - t_{k-3}.$$

if $3 \mid k+1$. Also, for the constants $D_{(a,b)}$, we define the sequence $(s_k)_k \in \mathbb{N} \cup \{0\}$ and $s_k := D_{(a,b)}$ if the shortest path from the root $(1, 1)$ to the ordered pair (a, b)

consists of k branches. By Corollary 5.3, this sequence is well-defined. We further claim that for all $k \geq 3$, we have

$$s_k = \frac{t_{k-2}s_{k-2}}{t_k} - \frac{s_{k-3}t_{k-3}}{t_k}$$

if $3 \nmid k+1$ and

$$s_k = \frac{t_{k-2}(2s_{k-2} - 3)}{t_k} - \frac{s_{k-3}t_{k-3}}{t_k}$$

if $3|k+1$. We prove both of these claims by induction.

We apply induction on the number of branches in the shortest path from the root $(1, 1)$ to (a, b) . Theorem 1.2 and Propositions 7.1 and 7.2 provide the cases for $k = 0$, $k = 1$, and $k = 2$. Suppose the result holds for pairs with shortest paths with $k-2$ branches and $k-3$ branches for some $k \geq 3$ and we want to show it also holds for k . Let (a, b) be a pair with a shortest path of n branches and let the fourth last and third last branches in this path have the pairs (a_0, a_1) and $(a_1, |a-b|)$ respectively. By our inductive hypothesis, we have

$$A_{(a_0, a_1)} = \frac{C_{(a_0, a_1)} \cdot 6.75^n}{n^{3/2}} \left(1 + \frac{s_{k-3}}{n} + O\left(\frac{1}{n^2}\right) \right)$$

and

$$A_{(a_1, |a-b|)} = \frac{C_{(a_1, |a-b|)} \cdot 6.75^n}{n^{3/2}} \left(1 + \frac{s_{k-2}}{n} + O\left(\frac{1}{n^2}\right) \right)$$

where

$$C_{(a_0, a_1)} = \frac{243 \cdot t_{k-3}}{4\sqrt{3}\pi}$$

and

$$C_{(a_1, |a-b|)} = \frac{243 \cdot t_{k-2}}{4\sqrt{3}\pi}.$$

Suppose first that $3 \nmid n+1$. Then we have either a and b are both odd or a is odd and b is even. By Lemma 5.7, we have for all $n \geq \lfloor \frac{k}{3} \rfloor$

$$A_{(a,b)}(n) = A_{(a_1, |a-b|)}(n) - A_{(a_0, a_1)}(n).$$

Then we have

$$\begin{aligned} A_{(a,b)}(n) &= \frac{C_{(a_1, |a-b|)} \cdot 6.75^n}{n^{3/2}} \left(1 + \frac{s_{k-2}}{n} + O\left(\frac{1}{n^2}\right) \right) - \frac{C_{(a_0, a_1)} \cdot 6.75^n}{n^{3/2}} \left(1 + \frac{s_{k-3}}{n} + O\left(\frac{1}{n^2}\right) \right) \\ &= \frac{(C_{(a_1, |a-b|)} - C_{(a_0, a_1)}) \cdot 6.75^n}{n^{3/2}} \left(1 + \left(\frac{t_{k-2}s_{k-2}}{t_k} - \frac{s_{k-3}t_{k-3}}{t_k} \right) \cdot \frac{1}{n} + O\left(\frac{1}{n^2}\right) \right) \\ &= \frac{(t_{k-2} - t_{k-3}) \cdot 6.75^n}{4\sqrt{3}\pi n^{3/2}} \left(1 + \left(\frac{t_{k-2}s_{k-2}}{t_k} - \frac{s_{k-3}t_{k-3}}{t_k} \right) \cdot \frac{1}{n} + O\left(\frac{1}{n^2}\right) \right). \end{aligned}$$

Thus $t_k - t_{k-2} = t_{k-3}$ and

$$s_k = \frac{t_{k-2}s_{k-2}}{t_k} - \frac{s_{k-3}t_{k-3}}{t_k}.$$

The case when $3|n+1$ is similar.

This proves our claims. Let $a_k = s_k t_k$. We can verify that

$$\lim_{k \rightarrow \infty} t_k = 0$$

so that for k sufficiently large we have by our recursive formulas for s_k that

$$a_k \approx a_{k-2} - a_{k-3}.$$

Solving this recurrence relation in much the same way we solved the recurrence relation in Theorem 1.1 gives the asymptotic

$$a_k \approx C \cdot (-1.3247\dots)^k$$

for some constant C where $-1.3247\dots$ is the only real root of

$$x^3 - x - 1.$$

Thus we obtain

$$s_k \approx \frac{C' \cdot (-2.6494\dots)^k}{k}$$

where C' depends on $k \bmod 3$. □

8. FURTHER QUESTIONS

On counting the number of (a, b) pairs in the Fibonacci Tree, there are still alot of questions that have been left unanswered. Some of these are as follows. Can we get even tighter bounds for $A(n)$? Theorem 1.2 above was essentially derived from Robbins' bounds for factorials. Since Robbins, however, there have been numerous improvements on bounds for factorials that will probably help us derive even better bounds for $A(n)$. For example, Knopp [4] shows that there exists constants A, B, C, D, \dots such that the sequence

$$r_n := \ln \left(\frac{n!e^n}{\sqrt{2\pi n}^{n+1/2}} \right)$$

is bounded above and below by the partial sums of

$$\frac{A}{n} - \frac{B}{n^3} + \frac{C}{n^5} - \frac{D}{n^7} + \dots$$

and Impens [3] shows how to compute those constants recursively. We may be able to use these results to prove that there exists positive constants A, B, C, D, \dots such that

$$\frac{A(n) \cdot 4\sqrt{3\pi n}^{3/2}}{243 \cdot 6.75^n}$$

can be approximated by

$$A + \frac{B}{n} + \frac{C}{n^2} + \frac{D}{n^3} + \dots$$

In this paper, we showed that $A = 1$, $B = \frac{-1387}{72}$ and that, if C exists, then $0 \leq C \leq \frac{5548}{9}$. We may be able to use the same procedure as in this paper to derive more terms of this series. Analogous questions remain open for $A_{(a,b)}$ for all coprime ordered pairs (a, b) .

We can also look at various of the Fibonacci Tree. For example, in taking a left branch from the ordered pair (x, y) do a subtraction $x - y$ instead of taking the mere difference $|x - y|$ or more generally for some $k \in \mathbb{N}$, take k children all of them being $x + \delta y$ where δ is a different k th root of unity for each one.

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APPENDIX A. TECHNICAL PROOFS

A.1. Algebra in Proof of Proposition 6.3.

A.1.1. *Binomial Theorem Calculations.* We have

$$\begin{aligned}
 \left(\frac{n+1}{n}\right)^{3/2} &= \left(\frac{n}{n+1}\right)^{-3/2} \\
 (A.1) \qquad \qquad \qquad &= \left(1 - \frac{1}{n+1}\right)^{-3/2}
 \end{aligned}$$

For all $0 < x \leq \frac{1}{101}$, we have, by the binomial theorem,

$$(A.2) \qquad (1-x)^{-3/2} = 1 + \frac{3x}{2} + \frac{15x^2}{8} + \dots + \frac{\frac{3}{2} \cdot \frac{5}{2} \cdot \dots \cdot \frac{2k+1}{2} x^k}{k!} + \dots$$

$$(A.3) \qquad \qquad \qquad < 1 + \frac{3x}{2} + \frac{19x^2}{10}.$$

Let $f(x) = (x+1)^{-3/2}$ and $g(x) = 1 - \frac{3x}{2}$. We have $f(0) = g(0) = 1$ and for all $x > 0$, we have

$$f'(x) = -\frac{3}{2}(x+1)^{-5/2} > -\frac{3}{2} = g'(x)$$

where $f(x)$ and $g'(x)$ are the derivatives of $f(x)$ and $g(x)$ respectively. Thus for all $x > 0$, we must have $f(x) > g(x)$ or

$$(A.4) \qquad \qquad \qquad (x+1)^{-3/2} > 1 - \frac{3x}{2}.$$

Thus we have

$$\begin{aligned}
D(n+1) &> \left(-\frac{405\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 + \frac{3}{2(n+1)} + \frac{19}{10(n+1)^2} \right) \left(1 - \frac{C}{(n+1)} \right) \\
&\quad + \frac{25 \cdot 6.75^{n+2}}{12\sqrt{3\pi}(n+1)^{3/2}} \left(1 - \frac{3}{2(n+1)} \right) \left(1 - \frac{23}{360(n+1)} \right) \\
&= \left(-\frac{405\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 + \frac{3-2C}{2(n+1)} + \frac{19-15C}{10(n+1)^2} - \frac{19C}{10(n+1)^3} \right) \\
&\quad + \frac{25 \cdot 6.75^{n+2}}{12\sqrt{3\pi}(n+1)^{3/2}} \left(1 - \frac{563}{360(n+1)} + \frac{11}{48(n+1)^2} - \frac{1}{5(n+1)^3} \right) \\
&> \left(-\frac{405\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 + \frac{3-2C}{2(n+1)} \right) \\
&\quad + \frac{25 \cdot 6.75^{n+2}}{12\sqrt{3\pi}(n+1)^{3/2}} \left(1 - \frac{563}{360(n+1)} \right).
\end{aligned}$$

A.1.2. *Proof that $r > 1$.* We have

$$\begin{aligned}
r &= \frac{32}{27} - \frac{4019}{1944C} \\
&> \frac{32}{27} - \frac{360}{1944} \\
&= 1.
\end{aligned}$$

A.1.3. *Proof of $D(n+k) > Inequality$.* We have

$$D(n+1) = \left(-\frac{405\sqrt{3} \cdot 6.75^{n+1}}{16\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 - \frac{C_1}{(n+1)} \right)$$

where $C_1 > rC$. Repeating the argument with C_1 in place of C and $n+1$ in place of n gives us

$$D(n+2) = \left(-\frac{405\sqrt{3} \cdot 6.75^{n+2}}{16\sqrt{\pi}(n+2)^{3/2}} \right) \left(1 - \frac{C_2}{(n+2)} \right)$$

where $C_2 > r_1 C_1$ where

$$r_1 := \frac{32}{27} - \frac{4019}{1944C_1} > \frac{32}{27} - \frac{4019}{1944C} = r$$

so that $C_2 > rC_1 > r^2C$. Repeating the argument as many times as necessary, we thus have, for all $k \in \mathbb{N}$,

$$D(n+k) = \left(-\frac{405\sqrt{3} \cdot 6.75^{n+k}}{16\sqrt{\pi}(n+k)^{3/2}} \right) \left(1 - \frac{C_k}{(n+k)} \right)$$

where $C_k > r^k C$.

A.1.4. *Proof of Second Inequality.* We can derive that

$$(A.5) \quad \frac{4019}{360n} < \frac{4019}{360(n+1)} + \frac{C}{32(n+1)}$$

and

$$(A.6) \quad \frac{448301}{(n+1)19440} < \frac{C}{27}$$

By Proposition 6.2, we have

$$\begin{aligned} D(n+1) &< \left(-\frac{405\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi}(n+1)^{3/2}} \right) \left(\frac{n+1}{n} \right)^{3/2} \left(1 - \frac{4019}{360n} + \frac{C}{n} \right) \\ &\quad + \frac{25 \cdot 6.75^{n+2}}{12\sqrt{3\pi}(n+1)^{3/2}} \left(\frac{n+1}{n+2} \right)^{3/2} \left(1 - \frac{23}{360(n+2)} + \frac{23}{25(n+2)^2} \right). \end{aligned}$$

By (A.1), (A.2), and (A.5), we have

$$\begin{aligned} D(n+1) &< \left(-\frac{405\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 + \frac{3}{2(n+1)} \right) \left(1 - \frac{4019}{360(n+1)} + \frac{C}{(n+1)} - \frac{C}{32(n+1)} \right) \\ &\quad + \frac{25 \cdot 6.75^{n+2}}{12\sqrt{3\pi}(n+1)^{3/2}} \left(1 + \frac{1}{(n+1)} \right)^{-3/2} \left(1 - \frac{23}{360(n+1)} + \frac{23}{360(n+1)^2} + \frac{23}{25(n+1)^2} \right) \end{aligned}$$

For all $0 < x < \frac{8}{9}$, we have, by the binomial theorem,

$$(1+x)^{-3/2} = 1 - \frac{3x}{2} + \frac{15x^2}{8} - \dots + \frac{(-1)^k \frac{3}{2} \cdot \frac{5}{2} \cdot \dots \cdot \frac{2k+1}{k} x^k}{k!}.$$

For all $k \geq 3$, k odd, we have

$$\begin{aligned} \frac{(-1)^k \frac{3}{2} \cdot \frac{5}{2} \cdot \dots \cdot \frac{2k+1}{k} x^k}{k!} + \frac{(-1)^{k+1} \frac{3}{2} \cdot \frac{5}{2} \cdot \dots \cdot \frac{2k+1}{k} x^{k+1}}{(k+1)!} &= -\frac{\frac{3}{2} \cdot \frac{5}{2} \cdot \dots \cdot \frac{2k+1}{k} x^k}{k!} + \frac{\frac{3}{2} \cdot \frac{5}{2} \cdot \dots \cdot \frac{2k+3}{2} x^{k+1}}{(k+1)!} \\ &= \frac{\frac{3}{2} \cdot \frac{5}{2} \cdot \dots \cdot \frac{2k+1}{k} x^k}{k!} \left(-1 + \frac{(2k+3)x}{2k+2} \right) \\ &\leq \frac{\frac{3}{2} \cdot \frac{5}{2} \cdot \dots \cdot \frac{2k+1}{k} x^k}{k!} \left(-1 + \frac{9x}{8} \right) \\ &< 0. \end{aligned}$$

Thus

$$(A.7) \quad (1+x)^{-3/2} < 1 - \frac{3x}{2} + \frac{15x^2}{8}.$$

Thus

$$\begin{aligned}
D(n+1) &< \left(-\frac{405\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 + \frac{3}{2(n+1)} \right) \left(1 - \frac{4019}{360(n+1)} + \frac{C}{(n+1)} - \frac{C}{32(n+1)} \right) \\
&\quad + \frac{25 \cdot 6.75^{n+2}}{12\sqrt{3\pi}(n+1)^{3/2}} \left(1 - \frac{3}{2(n+1)} + \frac{15}{8(n+1)^2} \right) \left(1 - \frac{23}{360(n+1)} + \frac{23}{360(n+1)^2} + \frac{23}{25(n+1)^2} \right) \\
&< \left(-\frac{405\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 - \frac{4019}{360(n+1)} + \frac{31C+48}{32(n+1)} + \frac{1395C-18236}{960(n+1)^2} \right) \\
&\quad + \frac{25 \cdot 6.75^{n+2}}{12\sqrt{3\pi}(n+1)^{3/2}} \left(1 - \frac{563}{360(n+1)} + \frac{10637}{3600(n+1)^2} - \frac{2553}{1600(n+1)^3} + \frac{1771}{960(n+1)^4} \right) \\
&< \left(-\frac{405\sqrt{3} \cdot 6.75^n}{2\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 - \frac{4019}{360(n+1)} + \frac{31C+48}{32(n+1)} + \frac{1395C-18236}{960(n+1)^2} \right) \\
&\quad + \frac{25 \cdot 6.75^{n+2}}{12\sqrt{3\pi}(n+1)^{3/2}} \left(1 - \frac{563}{360(n+1)} + \frac{10637}{3600(n+1)^2} \right) \\
&= \left(-\frac{405\sqrt{3} \cdot 6.75^{n+1}}{16\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 - \frac{4019}{360(n+1)} + \frac{31C}{27(n+1)} + \frac{33480C-448301}{19440(n+1)^2} \right).
\end{aligned}$$

By (A.6), we have

$$\begin{aligned}
D(n+1) &< \left(-\frac{405\sqrt{3} \cdot 6.75^{n+1}}{16\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 - \frac{4019}{360(n+1)} + \frac{31C}{27(n+1)} - \frac{448301}{19440(n+1)^2} \right) \\
&< \left(-\frac{405\sqrt{3} \cdot 6.75^{n+1}}{16\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 - \frac{4019}{360(n+1)} + \frac{31C}{27(n+1)} - \frac{C}{27(n+1)} \right) \\
&= \left(-\frac{405\sqrt{3} \cdot 6.75^{n+1}}{16\sqrt{\pi}(n+1)^{3/2}} \right) \left(1 - \frac{4019}{360(n+1)} + \frac{10C}{9(n+1)} \right).
\end{aligned}$$

From the fact that $\frac{16072}{45n} \leq C$ we can deduce that $\frac{16072}{(n+1)} \leq \frac{10C}{9}$. Thus we can repeat the above argument with $\frac{10C}{9}$ in place of C and $n+1$ in place of n to derive that

$$D(n+2) < \left(-\frac{405\sqrt{3} \cdot 6.75^{n+2}}{16\sqrt{\pi}(n+2)^{3/2}} \right) \left(1 - \frac{4019}{360(n+2)} + \frac{\left(\frac{10}{9}\right)^2 C}{(n+2)} \right).$$

Repeating the argument as many times as necessary, we get that, for all $k \in \mathbb{N}$,

$$D(n+k) < \left(-\frac{405\sqrt{3} \cdot 6.75^{n+k}}{16\sqrt{\pi}(n+k)^{3/2}} \right) \left(1 - \frac{4019}{360(n+k)} + \frac{\left(\frac{10}{9}\right)^k C}{(n+k)} \right).$$

We know that

$$\lim_{k \rightarrow \infty} \frac{-D(n+k)16\sqrt{\pi}(n+k)^{3/2}}{405\sqrt{3} \cdot 6.75^{n+k}} = 1$$

so that

$$\lim_{k \rightarrow \infty} \frac{-4019}{360(n+k)} - \frac{\left(\frac{10}{9}\right)^k C}{n+k} = 0.$$

Thus

$$\lim_{k \rightarrow \infty} \frac{\left(\frac{10}{9}\right)^k}{n+k} = 0$$

so that

$$\lim_{k \rightarrow \infty} \frac{\left(\frac{10}{9}\right)^k}{k} = 0,$$

which doesn't hold, a contradiction. Thus we have our second desired inequality for all $n \geq 100$.

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